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ON ANALYTIC CONTINUABILITY OF THE MISSING CAUCHY DATUM FOR HELMHOLTZ BOUNDARY PROBLEMS

MIRZA KARAMEHMEDOVIĆ

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ABSTRACT. We relate the domains of analytic continuation of Dirichlet and Neumann boundary data for Helmholtz problems in two or more independent variables. The domains are related *à priori*, locally and explicitly in terms of complex polyrectangular neighbourhoods of planar pieces of the boundary. To this end we identify and characterise a special subspace of the standard pseudodifferential operators with real-analytic symbols. The result is applicable in the estimation of the domain of analytic continuation of solutions across planar pieces of the boundary.

1. INTRODUCTION

For a second-order elliptic boundary problem in two or more independent variables, with only the Dirichlet or the Neumann condition specified, to what subset of the complexified boundary can the missing Cauchy datum be continued analytically? In view of existence results such as the theorem of Cauchy-Kovalevsky [5, Theorem 9.4.5], or the propagation of singularities of solutions to the analytic Cauchy problem [24], this question is central in the *à priori* estimation of the domain of analytic continuability of solutions across the boundary. This, in turn, is applicable, e.g., in the stability and convergence analysis of 'interior source methods', which is a family of promising numerical methods for direct and inverse elliptic boundary problems [2, 3, 8, 9, 10, 11].

In Millar [14, 15, 16, 17, 18, 19, 20, 21, 22], boundary integral representations of the solution are used to relate the Cauchy data, and the domain of analytic continuability of the missing boundary datum is estimated globally, essentially by analytic continuation of the integrals to complex space. This is done for general linear, elliptic, second-order, analytic, exterior boundary problems in two independent variables and with piecewise analytic boundary in [18], for such interior and exterior problems with analytic boundary in [21], as well as for exterior three-dimensional Helmholtz problems in a half-space [19] or with axisymmetric boundary [20]. Another global approach can be found in Section 4 of Sternin and Shatalov [25], for three-dimensional Helmholtz problems with Neumann datum given on an algebraic surface; see Sternin and Shatalov [24] for a more general treatment. More recently, Kangro, Kangro and Nicolaides [23] proposed a *local* approach to the *à priori* analytic continuation of solutions of two-dimensional Dirichlet boundary problems for the Helmholtz equation across analytic pieces of the boundary. Implicit in the method (see the comment immediately after Lemma 1 on p. 594 of [23]) is a regularity result regarding the missing Neumann datum, given essentially as follows: if both a parametrisation

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of the considered piece of the boundary and the (localised) Dirichlet datum can be analytically continued to a complex rectangle $\{t + i\eta \in \mathbb{C}, t \in [a, b], \eta \in [-c, c]\}$, then so can the missing (localised) Neumann datum. Kangro and Kangro [7] suggest a method analogous to that of [23] to handle *locally* the analytic continuation of solutions to three-dimensional Dirichlet problems for the Helmholtz equation across planar pieces of the boundary. There, they propose that if the Dirichlet datum can be analytically continued to the complex polyrectangle

$$\{(x + i\xi_1, y + i\xi_2), x \in [a_1 - 2r, a_1 + 2r], y \in [a_2 - 2r, a_2 + 2r], \xi_1, \xi_2 \in [-r, r]\}$$

in \mathbb{C}^2 , then the missing Neumann datum is analytically continuable to the set

$$\{(x + i\xi_1, y + i\xi_2), x \in [a_1 - r, a_1 + r], y \in [a_2 - r, a_2 + r], \xi_1^2 + \xi_2^2 \leq r^2\}.$$

The method of [23, 7] is, in principle, extendible to higher dimensions. However, as stated on p. 592 of [23], '[...] in more than two dimensions the computations become quite involved.'

We here estimate the domain of analytic continuation of the missing Cauchy datum on open planar subsets of the boundary for Dirichlet and Neumann problems for the Helmholtz equation in $n + 1$ independent variables, *for any* $n \in \mathbb{N}$. The analysis is *a priori* (that is, it does not require a solution of the boundary problem) and *local*, and in particular it uses no information about the boundary or about the Cauchy data outside the planar subset. To prove the main result, we identify and characterise a special subspace of the standard pseudodifferential operators on $\mathbb{R}^n \times \mathbb{R}^n$. The standard pseudodifferential operators are described, e.g., in Chapter XVIII of Hörmander [6]. The remainder of this section contains some notational conventions, a precise statement of the main result, and an overview of the rest of the paper.

In the following, n is a fixed positive integer. A function f defined on an open subset Ω of \mathbb{R}^n , or of \mathbb{C}^n , is here said to be real-analytic in Ω , respectively analytic in Ω , if for every point x in Ω there is a nonempty neighbourhood of x (in \mathbb{R}^n or \mathbb{C}^n , respectively,) in which the Taylor series of f about x is convergent and agrees with f . Elements z of \mathbb{C}^n are written (z_1, \dots, z_n) , and we adopt the notation $z_{(j)} = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$. Multiindices are understood to be n -tuples of nonnegative integers. For every multiindex α , the convention is that $|\alpha| = \sum \alpha_j$. The operator Δ is the Laplacian on \mathbb{R}^{n+1} , $\Delta = \sum_{j=1}^{n+1} \partial_j^2$, and $\Delta + k^2$ is the Helmholtz operator on \mathbb{R}^{n+1} . The constant k is assumed positive. Finally, $\mathcal{L}(A, B)$ is the vector space of continuous linear maps from A to B .

Now fix n -tuples $a, b^-, b^+ \in]0, \infty]^n$, write $\tau(a) = \prod_{j=1}^n] - a_j, a_j[$, and define the open complex polyrectangle

$$T(a, b^\pm) = \left\{ z \in \mathbb{C}^n, \operatorname{Re} z \in \tau(a), \operatorname{Im} z \in \prod_{j=1}^n] - b_j^-, b_j^+ [\right\}.$$

Fix $l \in \{0, 1\}$ and $\lambda > 0$, and assume $u \in C^1(\overline{\tau(a)} \times [0, \lambda])$ satisfies

$$(\Delta + k^2)u = 0 \quad \text{in } \tau(a) \times]0, \lambda[;$$

$$\lim_{x_j \rightarrow \pm a_j} \partial_j^r u(\cdot, 0) \text{ analytically continuable to } T(a_{(j)}, b_{(j)}^\pm) \text{ and in } C^1(\overline{T(a_{(j)}, b_{(j)}^\pm)})$$

$$\text{for } r = 0, 1, j = 1, \dots, n;$$

$$\lim_{x_{n+1} \nearrow \lambda} \partial_{n+1}^r u \text{ analytically continuable to } T(a, b^\pm) \text{ and in } C^1(\overline{T(a, b^\pm)}), \quad r = 0, 1;$$

$$U_l := \lim_{x_{n+1} \searrow 0} \partial_{n+1}^l u \text{ analytically continuable to } T(a, b^\pm) \text{ and in } C^1(\overline{T(a, b^\pm)}).$$

The C^1 regularity above is meant as one-sided, from the interior of the domain, and in the case of traces of the solution u , it is understood w.r.t. the variables $\operatorname{Re} z_j, \operatorname{Im} z_j$. The function U_l is the given boundary datum on $\tau(a)$. The main result here is the following.

Theorem 1.1. *The missing boundary datum $U_{1-l} = \lim_{x_{n+1} \searrow 0} \partial_{n+1}^{1-l} u$ is analytically continuable to each complex polyrectangle $T(\tilde{a}, \tilde{b}^\pm)$ with*

$$\tilde{a}_j \in]0, a_j[\quad \text{and} \quad \tilde{b}_j^\pm \in]0, \min \{a_j - \tilde{a}_j, b_j^\pm\}[, \quad j = 1, \dots, n.$$

The Helmholtz operator is invariant under translation and rotation in \mathbb{R}^{n+1} . Therefore, the setup of Theorem 1.1 can, without loss of generality, be understood as the localisation of a Helmholtz boundary problem about a point of the boundary where the latter is a subset of a hyperplane. The functions U_0 and U_1 are localisations of the corresponding Dirichlet and Neumann boundary datum, respectively.

The rest of this paper is organised as follows. In Section 2, we show a mapping property of a subspace of the standard pseudodifferential operators that are specially suited for a proof of Theorem 1.1. That section is inspired in part by the analysis of Boutet de Monvel [1]. Section 3 contains a proof of Theorem 1.1 and involves the development on pp. 109–110 of Section 18.2 and pp. 232–236 of Section 20.1 in Hörmander [6]. Finally, in Section 4, we comment on the generalisation of Theorem 1.1.

2. A MAPPING PROPERTY OF CERTAIN STANDARD PSEUDODIFFERENTIAL OPERATORS

Throughout this section, m is a real.

2.1. The space $S_{r-a}^{m,\varepsilon}$ of real-analytic symbols. The pseudodifferential symbols used in the proof of Theorem 1.1 are functions of only the cotangent variables in $T^*\mathbb{R}^n$. Recall that a function p is in the standard symbol space $S^m(\mathbb{R}^n)$ if and only if $p \in C^\infty(\mathbb{R}^n)$ and for each multiindex $\alpha \in \mathbb{N}_0^n$ there is a constant C_α satisfying

$$(2.1) \quad |\partial^\alpha p(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}, \quad \xi \in \mathbb{R}^n.$$

In the following, we write S^m for $S^m(\mathbb{R}^n)$. Also, $S^{-\infty} = \bigcap_{\mu \in \mathbb{R}} S^\mu$. With $\alpha \in \mathbb{N}_0^n$, the map taking each $p \in S^m$ to the smallest number C_α that satisfies (2.1) is a seminorm on S^m ; write $\|\cdot\|_\alpha^{(m)}$ for this seminorm. The space S^m is now equipped with the corresponding 'natural topology' \mathfrak{T}^m : the coarsest topology on S^m in which all the seminorms $\|\cdot\|_\alpha^{(m)}$, $\alpha \in \mathbb{N}_0^n$, are continuous, and in which the addition is continuous. Chapter XVIII of Hörmander [6] contains a treatment of the standard pseudodifferential symbols and operators.

Fix $R \geq 0$ and $\varepsilon \in]0, 1[$, and let

$$K_\varepsilon = \{\zeta \in \mathbb{C}, |\operatorname{Im} \zeta| < \varepsilon |\operatorname{Re} \zeta| \text{ if } |\operatorname{Re} \zeta| > R, \operatorname{Im} \zeta = 0 \text{ otherwise}\}.$$

That a function p is 'analytic on K_ε^n ' means in the following that for any $j \in \{1, \dots, n\}$ and any $\zeta_{(j)} = (\zeta_1, \dots, \zeta_{j-1}, \zeta_{j+1}, \dots, \zeta_n) \in K_\varepsilon^{n-1}$, the function $\zeta_j \mapsto p(\zeta_1, \dots, \zeta_n)$ is analytic on the interior K_ε° .

Definition 2.1. A function p is an element of the set $S_{r-a}^{m,\varepsilon}$ if $p \in S^m$, p is analytic on K_ε^n , there is a nonnegative C such that

$$(2.2) \quad |p(\zeta)| \leq C(1 + |\operatorname{Re} \zeta|)^m, \quad \zeta \in K_\varepsilon^n,$$

and $\sum_{\alpha \in \mathbb{N}_0^n} \|p\|_\alpha^{(m)} \varepsilon^{|\alpha|} / \alpha! < \infty$. Also, $S_{r-a}^{-\infty, \varepsilon} = \bigcap_{\mu \in \mathbb{R}} S_{r-a}^{\mu, \varepsilon}$.

$S_{r-a}^{m, \varepsilon}$ is a vector subspace of the space S^m of standard pseudodifferential symbols. The map taking each $p \in S_{r-a}^{m, \varepsilon}$ to the smallest number C satisfying (2.2) is a norm on $S_{r-a}^{m, \varepsilon}$, write $\|\cdot\|_*^{(m)}$ for this norm and \mathfrak{T}_*^m for the corresponding open ball topology on $S_{r-a}^{m, \varepsilon}$. The natural topology $\mathfrak{T}_{\text{rel}}^m$ on $S_{r-a}^{m, \varepsilon}$ is that induced by \mathfrak{T}^m , and it is in particular generated by the sets

$$\{p \in S_{r-a}^{m, \varepsilon}, \|p - q\|_\alpha^{(m)} < t\}, \quad \alpha \in \mathbb{N}_0^n, t > 0, q \in S^m.$$

Lemma 2.2. $\mathfrak{T}_{\text{rel}}^m$ is stronger than \mathfrak{T}_*^m .

Proof. Fix $p \in S_{r-a}^{m, \varepsilon}$ and $\delta > 0$. Given $\zeta \in K_\varepsilon^n$ with $\zeta_j \in [-R, R]$ for $j \in I \subseteq \{1, \dots, n\}$, and $\text{Re } \zeta_j > R$ for $j \in J = \{1, \dots, n\} \setminus I$, Taylor's formula implies, with $\xi = \text{Re } \zeta$ and $\eta = \text{Im } \zeta$,

$$|p(\zeta)| = \left| \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_{\xi, J}^\alpha p(\xi) (i\eta_J)^\alpha \right| \leq \sum_{\alpha \in \mathbb{N}_0^n} \frac{\|p\|_\alpha^{(m)}}{\alpha!} (1 + |\xi|)^{m-|\alpha|} \varepsilon^{|\alpha|} (1 + |\xi|)^{|\alpha|}.$$

Since $\sum_{\alpha \in \mathbb{N}_0^n} \|p\|_\alpha^{(m)} \varepsilon^{|\alpha|} / \alpha! = \sum_{|\alpha| \leq \mu} \|p\|_\alpha^{(m)} \varepsilon^{|\alpha|} / \alpha! + \sum_{|\alpha| > \mu} \|p\|_\alpha^{(m)} \varepsilon^{|\alpha|} / \alpha! < \infty$ for $\mu \in \mathbb{N}_0$, there is a natural μ and a positive t such that the open ball $\{\|p\|_*^{(m)} < \delta\} \in \mathfrak{T}_*^m$ includes the set

$$\bigcap_{|\alpha| \leq \mu} \{\|p\|_\alpha^{(m)} < t\} \in \mathfrak{T}_{\text{rel}}^m.$$

□

2.2. Real-analytic operators. Here we use the symbols in $S_{r-a}^{m, \varepsilon}$ to define a family of operators. The operators are expressed in terms of integrals over certain complex contours in $\mathbb{C}^n \times \mathbb{C}^n$, described first.

Let a, a', b^-, b^+ and c be n -tuples of elements of $]0, \infty]$ such that $a'_j < c_j < a_j$ and $\max\{b_j^-, b_j^+\} < \varepsilon(c_j - a'_j)$ for each $j = 1, \dots, n$. Let χ_j be a 'window function' in $C_0^\infty(\mathbb{R})$ satisfying

$$\begin{aligned} \chi_j(t) &= 1, & |t| &\leq c_j, \\ \chi_j(t) &\in]0, 1[, & c_j < |t| < a_j, \\ \chi_j(t) &= 0, & |t| &\geq a_j. \end{aligned}$$

With R a positive constant, let $\psi \in C^\infty(\mathbb{R})$ be an even 'excision function' satisfying

$$\begin{aligned} \psi(t) &= 1, & |t| &\geq 2R, \\ \psi(t) &\in]0, 1[, & R < |t| < 2R, \\ \psi(t) &= 0, & |t| &\leq R. \end{aligned}$$

Also, let for $j = 1, \dots, n$ and for all real t and t' :

$$\begin{aligned} s_j(t, t') &= -\chi_j(t) \psi(t') b_j^- \text{sgn } t' \text{sgn } t', \\ \eta_j(t, t') &= (\chi_j(t) - 1) \psi(t') b_j^- \text{sgn } t' |t'| / (c_j - a'_j) \text{sgn } t. \end{aligned}$$

Fix an odd positive integer N , write

$$s(y, \xi) = (s_1(y_1, \xi_1), \dots, s_n(y_n, \xi_n)) \quad \text{and} \quad \eta(y, \xi) = (\eta_1(y_1, \xi_1), \dots, \eta_n(y_n, \xi_n))$$

for all y and ξ in \mathbb{R}^n , introduce the map

$$\sigma : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}^{2n}$$

by

$$\sigma(t, y, \xi) = (y + it^N s(y, \xi), \xi + it^N \eta(y, \xi)),$$

and define the corresponding family of complex contours by

$$\mathcal{C}_t = \sigma(t, \mathbb{R}^n \times \mathbb{R}^n), \quad t \in [0, 1].$$

As shown in the proof of Theorem 2.6 below, the mapping $\mathbb{R}^n \times \mathbb{R}^n \ni (y, \xi) \mapsto \sigma(t, y, \xi)$ is orientation-preserving for each fixed $t \in [0, 1]$. Every contour \mathcal{C}_t is in the following given the orientation induced by its parametrisation $\sigma(t, \cdot, \cdot)$.

Remark 2.3. We construct the contours \mathcal{C}_t such that the function $\mathbb{C}^{2n} \ni (w, \zeta) \mapsto |e^{i\zeta(z-w)}|$ is decreasing with respect to $|\xi_j| = |\operatorname{Re} \zeta_j|$ when $t \in]0, 1[$, $(w, \zeta) \in \mathcal{C}_t$, $z \in T(a', t^N b^\pm)$ and $|\xi_j| \geq 2R$. Indeed, we have in this case that $(y_j - \operatorname{Re} z_j) \operatorname{sgn} y_j / (c_j - a'_j) \geq 1$ when $\chi_j(y_j) - 1 \neq 0$, as well as that $\psi(\xi_j) = 1$, so

$$\begin{aligned} \operatorname{Re} i\zeta_j(z_j - w_j) &= (t^N s_j - \operatorname{Im} z_j)\xi_j + (y_j - \operatorname{Re} z_j)t^N \eta_j \\ &= -|\xi_j|(\operatorname{Im} z_j \operatorname{sgn} \xi_j + t^N \chi_j b_j^{-\operatorname{sgn} \xi_j}) + t^N |\xi_j| b_j^{-\operatorname{sgn} \xi_j} (\chi_j - 1) \frac{y_j - \operatorname{Re} z_j}{c_j - a'_j} \operatorname{sgn} y_j \\ &\leq -|\xi_j|(\operatorname{Im} z_j \operatorname{sgn} \xi_j + t^N b_j^{-\operatorname{sgn} \xi_j}), \end{aligned}$$

and it is readily verified that $\operatorname{Im} z_j \operatorname{sgn} \xi_j + t^N b_j^{-\operatorname{sgn} \xi_j} > 0$. Note that, furthermore, $\chi_j(y_j) = 1$ implies $\operatorname{Re} i\zeta_j(z_j - w_j) = -|\xi_j|(\operatorname{Im} z_j \operatorname{sgn} \xi_j + t^N b_j^{-\operatorname{sgn} \xi_j})$, which is negative only if $\operatorname{Im} z_j \in]-t^N b_j^-, t^N b_j^+[$. Finally, if $\chi_j(y_j) < 1$ and $\operatorname{Im} z_j = 0$, then $\operatorname{Re} i\zeta_j(z_j - w_j) = -|\xi_j| t^N b_j^{-\operatorname{sgn} \xi_j} (\chi_j + (1 - \chi_j)(y_j - z_j) \operatorname{sgn} y_j / (c_j - a'_j))$, which is negative if and only if $z_j \in]-c_j, c_j[$.

The real-analytic operators of this section are constructed to map to and from the following spaces of functions.

Definition 2.4. The space $\mathcal{O}(a', b^\pm)$ consists precisely of those functions that are analytic on $T(a', b^\pm)$. The elements of the space $\tilde{\mathcal{O}}(a, b^\pm)$ are functions that are analytic on $T(a, b^\pm)$, C^1 w.r.t. $\operatorname{Re} z_j$ and $\operatorname{Im} z_j$ on the closure $\bar{T}(a, b^\pm)$ and that vanish on $\mathbb{R}^n \setminus \tau(a)$.

We equip the space $\mathcal{O}(a', b^\pm)$ with the topology of uniform convergence on compact subsets of $T(a', b^\pm)$, and the space $\tilde{\mathcal{O}}(a, b^\pm)$ with the topology induced by the usual $C^{1, \alpha}$ norm

$$\|u\| = \sum_{\substack{\beta \in \mathbb{N}_0^{2n} \\ |\beta| \leq 1}} \left(\|\partial_{\operatorname{Re} z, \operatorname{Im} z}^\beta u\|_{L^\infty(\bar{T}(a, b^\pm))} + \sup_{T(a, b^\pm)} \frac{|\partial_{\operatorname{Re} z, \operatorname{Im} z}^\beta u(x) - \partial_{\operatorname{Re} z, \operatorname{Im} z}^\beta u(y)|}{|x - y|^\alpha} \right)$$

for some $\alpha \in]0, 1[$. It is well-known that $\mathcal{O}(a', b^\pm)$ is then Fréchet and $\tilde{\mathcal{O}}(a, b^\pm)$ is Banach. The spaces

$$\mathcal{L}(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(c))) \quad \text{and} \quad \mathcal{L}(\tilde{\mathcal{O}}(a, b^\pm), \mathcal{O}(a', b^\pm))$$

of continuous linear mappings are equipped with the corresponding topologies of bounded convergence and are thus made locally convex (see pp. 131-133 of Köthe [12]). In particular, the topologies on these spaces are defined by the systems of seminorms

$$\|Q\|_W = \sup_{u \in W} \|Qu\|_{L^\infty(\tau(c))}$$

and

$$\|Q\|_{W,\kappa} = \sup_{u \in W} \sup_{z \in \kappa} |Qu(z)|,$$

respectively, where W is a bounded subset of $\tilde{\mathcal{O}}(a, b^\pm)$ and κ is a compact subset of $T(a', b^\pm)$. Since $\tilde{\mathcal{O}}(a, b^\pm)$ is normed and $L^\infty(\tau(c))$ is Banach, $\mathcal{L}(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(c)))$ is complete.

Definition 2.5. For every symbol p in $S_{r-a}^{m,\varepsilon}$, $\text{OP}(p)$ is the operator on $\tilde{\mathcal{O}}(a, b^\pm)$ given by (2.3)

$$\text{OP}(p)u(z) = (2\pi)^{-n} \int_{(w,\zeta) \in \mathcal{C}_1} e^{i\zeta(z-w)} p(\zeta) u(w), \quad z \in T(a', b^\pm), u \in \tilde{\mathcal{O}}(a, b^\pm).$$

Theorem 2.6. *The maps*

$$\text{OP} : S_{r-a}^{m,\varepsilon} \rightarrow \mathcal{L}(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(c)))$$

and

$$\text{OP} : S_{r-a}^{m,\varepsilon} \rightarrow \mathcal{L}(\tilde{\mathcal{O}}(a, b^\pm), \mathcal{O}(a', b^\pm))$$

are linear and continuous when $S_{r-a}^{m,\varepsilon}$ is equipped with the natural topology $\mathfrak{T}_{\text{rel}}^m$.

Furthermore, if $p \in S_{r-a}^{-\infty,\varepsilon}$, $u \in \tilde{\mathcal{O}}(a, b^\pm)$ and $x \in \tau(a')$, then

$$(2.4) \quad \text{OP}(p)u(x) = (2\pi)^{-n} \int_{(y,\xi) \in \mathcal{C}_0} e^{i\xi(x-y)} p(\xi) u(y).$$

Proof. Let δ^{Kr} be the 'Kronecker delta'; that is, for all integers j, l set $\delta_{j,l}^{\text{Kr}} = 0$ if $j \neq l$ and $\delta_{j,l}^{\text{Kr}} = 1$ if $j = l$. For all $j, l = 1, \dots, n$ and all $(t, y, \xi) \in]0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$, we have

$$(2.5) \quad \begin{cases} \partial_t \sigma_l(t, y, \xi) &= -iNt^{N-1} \chi_l(y_l) \psi(\xi_l) b_l^{-\text{sgn } \xi_l} \text{sgn } \xi_l, \\ \partial_t \sigma_{l+n}(t, y, \xi) &= iNt^{N-1} (\chi_l(y_l) - 1) \psi(\xi_l) b_l^{-\text{sgn } \xi_l} |\xi_l| \text{sgn } y_l / (c_l - a'_l), \end{cases}$$

$$(2.6) \quad \begin{cases} \partial_{y_j} \sigma_l(t, y, \xi) &= \delta_{j,l}^{\text{Kr}} \left(1 - it^N \chi'_j(y_j) \psi(\xi_j) b_j^{-\text{sgn } \xi_j} \text{sgn } \xi_j \right), \\ \partial_{y_j} \sigma_{l+n}(t, y, \xi) &= it^N \delta_{j,l}^{\text{Kr}} \chi'_j(y_j) \psi(\xi_j) b_j^{-\text{sgn } \xi_j} |\xi_j| \text{sgn } y_j / (c_j - a'_j), \end{cases}$$

and

$$(2.7) \quad \begin{cases} \partial_{\xi_j} \sigma_l(t, y, \xi) &= -it^N \delta_{j,l}^{\text{Kr}} \chi_j(y_j) \psi'(\xi_j) b_j^{-\text{sgn } \xi_j} \text{sgn } \xi_j, \\ \partial_{\xi_j} \sigma_{l+n}(t, y, \xi) &= \delta_{j,l}^{\text{Kr}} \left(1 + it^N (\chi_j(y_j) - 1) b_j^{-\text{sgn } \xi_j} \text{sgn } y_j \frac{\psi'(\xi_j) |\xi_j| + \psi(\xi_j) \text{sgn } \xi_j}{c_j - a'_j} \right). \end{cases}$$

Thus, for each $t \in]0, 1]$, the Jacobi determinant of the mapping $(y, \xi) \mapsto \sigma(t, y, \xi)$ is at most polynomially increasing with respect to $|\xi_1|, \dots, |\xi_n|$, and, in view of Remark 2.3, the function $(w, \zeta) \mapsto e^{i\zeta(z-w)} p(\zeta) u(w)$ is absolutely integrable over the contour \mathcal{C}_t when $z \in \tau(c)$ or $z \in T(a', t^N b^\pm)$, as well as $t \in]0, 1]$, $p \in S_{r-a}^{m,\varepsilon}$ and $u \in \tilde{\mathcal{O}}(a, b^\pm)$. In particular, we have for all $p \in S_{r-a}^{m,\varepsilon}$ and $u \in \tilde{\mathcal{O}}(a, b^\pm)$ that

$$|\text{OP}(p)u(z)| \leq C(z) \|p\|_*^{(m)} \|u\|, \quad z \in \tau(c) \cup T(a', b^\pm),$$

where

$$C(z) = \int_{(y,\xi) \in \tau(a) \times \mathbb{R}^n} (1 + |\xi|)^m \left| e^{i\zeta(z-w)} \det d_{y,\xi} \sigma(1, y, \xi) \right|, \quad z \in \tau(c) \cup T(a', b^\pm),$$

is a continuous function independent of p and u (see part (a) of Theorem (2.27) on p. 54 in Folland [4]). Thus the linear map $\text{OP}(p) : \tilde{\mathcal{O}}(a, b^\pm) \rightarrow L^\infty(\tau(c))$ is continuous when $p \in S_{r-a}^{m,\varepsilon}$. Also, there is a nonnegative constant C such that, for all $p \in S_{r-a}^{m,\varepsilon}$ and every bounded subset W of $\tilde{\mathcal{O}}(a, b^\pm)$,

$$\sup_{u \in W} \|\text{OP}(p)u\|_{L^\infty(\tau(c))} \leq (C \sup_{u \in W} \|u\|) \|p\|_*^{(m)},$$

so the linear map $\text{OP} : S_{r-a}^{m,\varepsilon} \rightarrow \mathcal{L}(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(c)))$ is continuous w.r.t. the open ball topology \mathfrak{T}_*^m on $S_{r-a}^{m,\varepsilon}$, and hence it is also continuous w.r.t. the stronger topology $\mathfrak{T}_{\text{rel}}^m$.

For each $j = 1, \dots, n$ and each $z \in T(a', b^\pm)$ we have (see part (b) of Theorem (2.27) in [4])

$$(2\pi)^n \partial_{\bar{z}_j} \text{OP}(p)u(z) = \partial_{\bar{z}_j} \int_{\mathcal{C}_1} e^{i\zeta(z-w)} p(\zeta) u(w) = \int_{\mathcal{C}_1} \partial_{\bar{z}_j} e^{i\zeta(z-w)} p(\zeta) u(w) = 0.$$

Furthermore, for every $p \in S_{r-a}^{m,\varepsilon}$ and every compact subset κ of $T(a', b^\pm)$, there exists a positive C_κ , depending only on κ , such that

$$\sup_{z \in \kappa} |\text{OP}(p)u(z)| \leq (C_\kappa \|p\|_*^{(m)}) \|u\|$$

for all $u \in \tilde{\mathcal{O}}(a, b^\pm)$, and the linear map $\text{OP}(p) : \tilde{\mathcal{O}}(a, b^\pm) \rightarrow \mathcal{O}(a', b^\pm)$ is continuous. Finally, for every bounded subset W of $\tilde{\mathcal{O}}(a, b^\pm)$ and every compact subset κ of $T(a', b^\pm)$ we have

$$\sup_{u \in W} \sup_{z \in \kappa} |\text{OP}(p)u(z)| \leq (C_\kappa \sup_{u \in W} \|u\|) \|p\|_*^{(m)}, \quad p \in S_{r-a}^{m,\varepsilon},$$

so the linear map

$$\text{OP} : S_{r-a}^{m,\varepsilon} \rightarrow \mathcal{L}(\tilde{\mathcal{O}}(a, b^\pm), \mathcal{O}(a', b^\pm))$$

is continuous w.r.t. the topology \mathfrak{T}_*^m , and hence also w.r.t. $\mathfrak{T}_{\text{rel}}^m$. This finishes the proof of the first part of Theorem 2.6.

We now prove the validity of (2.4) using Stokes's theorem. Fix $\rho > R$ and write

$$D_\rho =]0, 1[\times \tau(a) \times (] - \rho, \rho[\setminus [-R, R]^n),$$

as well as $M_\rho = \sigma(D_\rho)$. The set M_ρ is included in $T(a, b^\pm) \times K_\varepsilon^n$, since for each $t \in]0, 1[$, $j = 1, \dots, n$, $y_j \in \mathbb{R}$ and $\xi_j \in \mathbb{R}$ we have $t^N s_j(y_j, \xi_j) \in [-t^N b_j^-, t^N b_j^+] \subset] - b_j^-, b_j^+ [$, as well as $|t^N \eta_j(y_j, \xi_j)| \leq t^N |\psi(\xi_j)| \cdot |\xi_j| b_j^{-\text{sgn} \xi_j} / (c_j - a_j')$, with $b_j^{-\text{sgn} \xi_j} / (c_j - a_j') < \varepsilon$. The closure \overline{D}_ρ is a compact, smooth and oriented $2n + 1$ -manifold with corners, and the map σ is smooth and injective on an open neighbourhood of \overline{D}_ρ . The injectivity follows from the fact that the functions s and η are real-valued, and N is odd. Indeed, if $y + it^N s(y, \xi) = \tilde{y} + it^N s(\tilde{y}, \tilde{\xi})$ and $\xi + it^N \eta(y, \xi) = \tilde{\xi} + it^N \eta(\tilde{y}, \tilde{\xi})$, then $y = \tilde{y}$ and $\xi = \tilde{\xi}$, and hence $t = \tilde{t}$. It is here of interest to study the rank of σ and of some of its restrictions. The matrices $(\text{Re } \partial_{y_j} \sigma_l)_{l,j}$ and $(\text{Re } \partial_{\xi_j} \sigma_{l+n})_{l,j}$ are the identity matrix, and the matrices $(\text{Re } \partial_{\xi_j} \sigma_l)_{l,j}$ and $(\text{Re } \partial_{y_j} \sigma_{l+n})_{l,j}$ are the zero matrix, so for every t in an open neighbourhood of $[0, 1]$ the Jacobian of the map $(y, \xi) \mapsto \sigma(t, y, \xi)$ has rank $2n$, and each contour $\mathcal{C}_{t,\rho} = \sigma(t, \tau(a) \times] - \rho, \rho[\setminus [-R, R]^n)$, $t \in [0, 1]$, is a $2n$ -dimensional immersed smooth submanifold of \mathbb{C}^{2n} . For each (t, y, ξ) in D_ρ , the quantity $\partial_t \sigma(t, y, \xi)$ is nonzero and purely imaginary, since $|\xi_l| > R$ for some $l = 1, \dots, n$, and since $\chi_l(y_l)$ and $\chi_l(y_l) - 1$ are never both zero. Thus, the Jacobian of the map σ has full rank on D_ρ , so σ is an

injective immersion there and the set M_ρ is a smooth, $(2n + 1)$ -dimensional immersed submanifold of \mathbb{C}^{2n} . Similarly, each element of the contour

$$\mathcal{D}_{j,\rho}^\pm = \sigma\left(]0, 1[\times \tau(a) \times \left\{ \xi_l \in]-\rho, \rho[\text{ for } l \neq j, \xi_j = \pm\rho \right\}\right), \quad j = 1, \dots, n,$$

is a $2n$ -dimensional immersed smooth submanifold of \mathbb{C}^{2n} .

The map σ induces an orientation on M_ρ , where the ordered basis of TM_ρ given by $(d\sigma \frac{d}{dt}, d\sigma \partial_{y_1}, \dots, d\sigma \partial_{y_n}, d\sigma \partial_{\xi_1}, \dots, d\sigma \partial_{\xi_n})$ is positively oriented. The restrictions of σ to the sets $\sigma^{-1}(\mathcal{C}_{1,\rho})$, $\sigma^{-1}(\mathcal{D}_{j,\rho}^+)$ with $j + n$ even, as well as $\sigma^{-1}(\mathcal{D}_{j,\rho}^-)$ with $j + n$ odd, preserve the Stokes orientation of the corresponding contours $\mathcal{C}_{1,\rho}$, $\mathcal{D}_{j,\rho}^+$ ($j + n$ even) and $\mathcal{D}_{j,\rho}^-$ ($j + n$ odd). The restrictions of σ to the sets $\sigma^{-1}(\mathcal{C}_{0,\rho})$, $\sigma^{-1}(\mathcal{D}_{j,\rho}^+)$ with $j + n$ odd, as well as $\sigma^{-1}(\mathcal{D}_{j,\rho}^-)$ with $j + n$ even, reverse the Stokes orientation of the corresponding contours $\mathcal{C}_{0,\rho}$, $\mathcal{D}_{j,\rho}^+$ ($j + n$ odd) and $\mathcal{D}_{j,\rho}^-$ ($j + n$ even). Furthermore, there is an open neighbourhood \mathcal{N} of $\overline{\tau(a)} \times ([-\rho, \rho]^n \setminus]-R, R[^n)$ such that, for all $j = 1, \dots, n$,

$$\begin{aligned} d\sigma_j &= iNt^{N-1}s_j dt + (1 + it^N \partial_{y_j} s_j) dy_j + it^N \partial_{\xi_j} s_j d\xi_j \quad \text{and} \\ d\sigma_{n+j} &= iNt^{N-1}\eta_j dt + it^N \partial_{y_j} \eta_j dy_j + (1 + it^N \partial_{\xi_j} \eta_j) d\xi_j \end{aligned}$$

when $(t, y, \xi) \in]0, 1[\times \mathcal{N}$. The restrictions of the forms dy_j and $d\xi_j$ to those subsets of ∂D_ρ where y_j or ξ_j is constant, respectively, are identically zero. Since $s_j(y_j, \xi_j)$ and $\partial_{\xi_j} s_j(y_j, \xi_j)$ equal zero when $|y_j| = a_j$, the restriction of $d\sigma_j$ to any subset of ∂D_ρ where $t \in]0, 1[$ and $|y_j| = a_j$ is identically zero. Also, since $\eta_j(y_j, \xi_j)$ and $\partial_{y_j} \eta_j(y_j, \xi_j)$ equal zero when $|\xi_j| = R$, the restriction of $d\sigma_{n+j}$ to any subset of ∂D_ρ where $t \in]0, 1[$ and $|\xi_j| = R$ is identically zero. Finally, each component of the boundary ∂D_ρ where two or more of the quantities $t, y_1, \dots, y_n, \xi_1, \dots, \xi_n$ are constant has dimension less than $2n$, so restrictions of differential $2n$ -forms to such components of ∂D_ρ are identically zero.

Now fix $p \in S_{r-a}^{-\infty, \varepsilon}$, $u \in \tilde{\mathcal{O}}(a, b^\pm)$ and $x \in \tau(a')$. The function

$$G(w, \zeta) = e^{i\zeta(x-w)} p(\zeta) u(w)$$

is analytic on $T(a, b^\pm) \times K_\varepsilon^n$, so the associated complex $2n$ -form

$$\mu = Gdw_1 \wedge \dots \wedge dw_n \wedge d\zeta_1 \wedge \dots \wedge d\zeta_n = Gdw \wedge d\zeta$$

is closed there:

$$d\mu = \sum_{j=1}^n \left(\partial_{w_j} Gdw_j + \partial_{\bar{w}_j} Gd\bar{w}_j + \partial_{\zeta_j} Gd\zeta_j + \partial_{\bar{\zeta}_j} Gd\bar{\zeta}_j \right) dw \wedge d\zeta = 0.$$

Also, the form $\sigma^* \mu = (G \circ \sigma) d\sigma_1 \wedge \dots \wedge d\sigma_{2n}$ is C^1 on the closure \overline{D}_ρ , so

$$(2.8) \quad \int_{M_\rho} d\mu = \int_{D_\rho} \sigma^* d\mu = \int_{D_\rho} d\sigma^* \mu = \int_{\partial D_\rho} \sigma^* \mu,$$

where the last equality follows from Stokes's theorem on manifolds with corners, given, e.g., as Theorem 14.20 on p. 367 in Lee [13]. Since $\mathcal{C}_{0,\rho}$ and $\mathcal{C}_{1,\rho}$ both include the set $\tau(a) \times [-R, R]^n$ but have mutually opposite Stokes orientation, (2.8) implies

$$(2.9) \quad \int_{\mathcal{C}_{1,\rho}} \mu - \int_{\mathcal{C}_{0,\rho}} \mu + \sum_{\substack{1 \leq j \leq n \\ j+n \text{ even}}} \left(\int_{\mathcal{D}_{j,\rho}^+} \mu - \int_{\mathcal{D}_{j,\rho}^-} \mu \right) + \sum_{\substack{1 \leq j \leq n \\ j+n \text{ odd}}} \left(\int_{\mathcal{D}_{j,\rho}^-} \mu - \int_{\mathcal{D}_{j,\rho}^+} \mu \right) = 0.$$

For $t \in \{0, 1\}$ we have

$$\begin{aligned} \int_{\mathcal{C}_{t,\rho}} \mu &= \int_{\mathcal{C}_{0,\rho}} \sigma(t, \cdot)^* \mu = \int_{(y,\xi) \in \mathcal{C}_{0,\rho}} G \circ \sigma(t, y, \xi) \det(d_{y,\xi} \sigma(t, y, \xi)) dy \wedge d\xi \\ &= \int_{(y,\xi) \in \mathcal{C}_{0,\rho}} G \circ \sigma(t, y, \xi) \det(d_{y,\xi} \sigma(t, y, \xi)) = \int_{\mathcal{C}_{t,\rho}} G. \end{aligned}$$

The function G is absolutely integrable over \mathcal{C}_0 since $\mathcal{C}_0 = \mathbb{R}^n \times \mathbb{R}^n$, $p \in S^{-\infty}$, and $u \in L^\infty(\mathbb{R}^n)$ is compactly supported. In view of the discussion near the beginning of this proof, the function G is thus absolutely integrable over each contour \mathcal{C}_t , $t \in [0, 1]$. Therefore, in particular,

$$\lim_{\rho \rightarrow \infty} \int_{\mathcal{C}_{0,\rho}} G = \int_{\mathcal{C}_0} G \quad \text{and} \quad \lim_{\rho \rightarrow \infty} \int_{\mathcal{C}_{1,\rho}} G = \int_{\mathcal{C}_1} G.$$

It remains to estimate the integrals over the contours $\mathcal{D}_{j,\rho}^\pm$ in (2.9). Using repeated integration by parts, we first note that, for each $\nu \in \mathbb{N}_0$ and $\rho > 1$,

$$\int_{t=0}^1 t^{N-1+\nu} e^{-t^N \rho b_j^{-\text{sgn } \xi_j}} = O(\rho^{-N-\nu}).$$

As seen from (2.5)–(2.7), the Jacobi determinant of the restriction of σ to $\sigma^{-1}(\mathcal{D}_{j,\rho}^\pm)$ is a polynomial in the variables $t, |\xi_1|, \dots, |\xi_{j-1}|, |\xi_{j+1}|, \dots, |\xi_n|$. The first column of the Jacobian is proportional to t^{N-1} , so each term of the determinant is of order at least $N-1$ w.r.t. t . Write M' for the maximal order of the Jacobi determinant w.r.t. the variables $(|\xi_1|, \dots, |\xi_{j-1}|, |\xi_{j+1}|, \dots, |\xi_n|)$. There are constants

$$M \in \mathbb{N}_0,$$

$$C', C'' \in \mathbb{R}, \text{ and}$$

$$C_{\nu,\alpha} \in \mathbb{R} \text{ for } \nu = 0, \dots, M, \alpha \in \mathbb{N}_0^{n-1}, |\alpha| \leq M',$$

such that for all $\rho > 1$

$$\begin{aligned} \left| \int_{\mathcal{D}_{j,\rho}^\pm} \mu \right| &\leq \sum_{\nu=0}^M \sum_{\substack{\alpha \in \mathbb{N}_0^{n-1} \\ |\alpha| \leq M'}} C_{\nu,\alpha} \rho^{|\alpha|} \int_{\sigma^{-1}(\mathcal{D}_{j,\rho}^\pm)} t^{N-1+\nu} |G \circ \sigma| \\ &\leq C' \|p\|_0^{(m)} \|u\| \rho^{M'} (1 + \rho \sqrt{n})^{|m|} \sum_{\nu=0}^M \int_{t=0}^1 t^{N-1+\nu} e^{-t^N \rho b_j^{-\text{sgn } \xi_j}} \\ &\quad \times \prod_{l \neq j} \int_{\xi_l = -\rho}^{\rho} e^{-t^N |\xi_l| b_l^{-\text{sgn } \xi_l} \psi(\xi_l)} \\ &\leq C'' \rho^{M'+|m|} \sum_{\nu=0}^M \int_{t=0}^1 t^{N-1+\nu} e^{-t^N \rho b_j^{-\text{sgn } \xi_j}} \cdot (2\rho)^{n-1} \\ &= O(\rho^{M'+|m|+n-1-N}), \end{aligned}$$

and, in conclusion, $\lim_{\rho \rightarrow \infty} \int_{\mathcal{D}_{j,\rho}^\pm} \mu = 0$ for sufficiently large N . This completes the proof of the second part of Theorem 2.6. \square

For $p \in S^{-\infty}$, the operator $P(p)$ is defined on $\tilde{\mathcal{O}}(a, b^\pm)$ by

$$P(p)u(x) = (2\pi)^{-n} \iint_{(y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n} e^{i\xi(x-y)} p(\xi) u(y), \quad x \in \mathbb{R}^n, u \in \tilde{\mathcal{O}}(a, b^\pm).$$

The mapping $P : S_{r-a}^{-\infty, \varepsilon} \rightarrow \mathcal{L} \left(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(a')) \right)$ is linear and continuous. Indeed, there is a nonnegative C such that

$$|P(p)u(x)| \leq \|p\|_0^{(-n-1)} \|u\| \int_{y \in \tau(a)} \int_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-n-1} \leq C \|p\|_0^{(-n-1)} \|u\|$$

for all $p \in S_{r-a}^{-\infty, \varepsilon}$, $u \in \tilde{\mathcal{O}}(a, b^\pm)$ and $x \in \tau(a')$.

Lemma 2.7. *OP is the unique extension of P that is linear and continuous as a map*

$$OP : S_{r-a}^{m, \varepsilon} \rightarrow \mathcal{L} \left(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(a')) \right)$$

for every fixed real m .

Proof. We first show the uniqueness of the extension. Pick $p \in S_{r-a}^{\mu, \varepsilon}$ for some real μ , and consider the family of functions $f_\lambda(\zeta) = e^{-(\lambda\zeta)^2}$, $\zeta \in K_\varepsilon^n$, with $\lambda \geq 0$. Each f_λ is analytic on K_ε^n . Also, $|f_\lambda(\zeta)| \leq \exp(-\lambda^2(1 - \varepsilon^2)|\operatorname{Re} \zeta|^2)$ on K_ε^n , so, for $\lambda > 0$, we have $f_\lambda \in S_{r-a}^{-\infty, \varepsilon}$ and $f_\lambda p \in S_{r-a}^{-\infty, \varepsilon}$. Since $f_1 \in S^0$, it follows from Proposition 18.1.2 on page 66 of Hörmander [6] that $\lim_{\lambda \searrow 0} f_\lambda = f_0 = 1$ in S^t for every positive t , so $\lim_{\lambda \searrow 0} f_\lambda p = p$ in $S^{\mu+t}$ for $t > 0$. Now if P'_1 and P'_2 are extensions of P as described above, then $(P'_1 - P'_2)p = (P'_1 - P'_2) \lim_{\lambda \searrow 0} f_\lambda p = \lim_{\lambda \searrow 0} (P'_1 - P'_2) f_\lambda p = 0$, where $\lim_{\lambda \searrow 0} f_\lambda p$ is understood w.r.t. the topology $\mathfrak{T}_{\text{rel}}^{\mu+t}$.

Next, $\tau(a') \subset \tau(c)$, so the first part of Theorem 2.6 implies that OP maps $S_{r-a}^{m, \varepsilon}$ continuously to $\mathcal{L} \left(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(a')) \right)$ for each real m . Finally, the second part of that theorem implies that $P(p)u = OP(p)u$ in $L^\infty(\tau(a'))$ when $p \in S_{r-a}^{-\infty, \varepsilon}$ and $u \in \tilde{\mathcal{O}}(a, b^\pm)$. \square

Write $OPS(p)$ for the standard pseudodifferential operator with symbol p , as defined in Section 18.1 of Hörmander [6]. Lemma 2.7 implies the following.

Corollary 2.8. *If $p \in S_{r-a}^{m, \varepsilon}$ and $u \in \tilde{\mathcal{O}}(a, b^\pm)$, then $OPS(p)u$ is the restriction to $\tau(a')$ of a function in $\mathcal{O}(a', b^\pm)$.*

Proof. The mapping OPS is by definition the unique linear continuous extension of the mapping $P : S_{r-a}^{-\infty, \varepsilon} \rightarrow \mathcal{L} \left(\tilde{\mathcal{O}}(a, b^\pm), L^\infty(\tau(a')) \right)$ to $S_{r-a}^{m, \varepsilon}$. Thus, by Lemma 2.7, we have for all $p \in S_{r-a}^{m, \varepsilon}$ and all $u \in \tilde{\mathcal{O}}(a, b^\pm)$ that $\sup_{x \in \tau(a')} |OPS(p)u(x) - OP(p)u(x)| = 0$. Finally, as shown in the first part of Theorem 2.6, $OP(p)u \in \mathcal{O}(a', b^\pm)$. \square

Now follows the main result of this section. Let f , g^- and g^+ be n -tuples of elements of $]0, \infty]$, and assume $p \in S_{r-a}^{m, \varepsilon}$.

Theorem 2.9. *If $u \in \tilde{\mathcal{O}}(f, g^\pm)$, then $OPS(p)u$ is analytically continuable to each complex polyrectangle $T(\tilde{a}, \tilde{b}^\pm)$ with $\tilde{a}_j \in]0, f_j[$ and $\tilde{b}_j^\pm \in]0, \varepsilon \min \{f_j - \tilde{a}_j, g_j^\pm\}[$, $j = 1, \dots, n$.*

Proof. The result follows by Corollary 2.8 and the fact that, in the definition of $OP(p)u$, each $c_j < f_j$ can be chosen arbitrarily close to f_j . \square

Remark 2.10. If m_j is real and $p_j \in S_{r-a}^{m_j, \varepsilon}$ for $j = 1, 2$, then the symbol of the compound $OPS(p_1)OPS(p_2)$ is in $S_{r-a}^{m_1+m_2, \varepsilon}$. Indeed, the symbols p_j only depend on the cotangent variable $\xi \in \mathbb{R}^n$, so the symbol of the compound $OPS(p_1)OPS(p_2)$ is the function $p_1 p_2$. By the standard calculus of pseudodifferential operators, $p_1 p_2 \in S^{m_1+m_2}$. The function $p_1 p_2$ is analytic on K_ε^n , and finally we have $|(p_1 p_2)(\zeta)| \leq \|p_1\|_*^{(m_1)} \|p_2\|_*^{(m_2)} (1 + |\operatorname{Re} \zeta|)^{m_1+m_2}$ for $\zeta \in K_\varepsilon^n$.

3. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we first relate the Cauchy data

$$U_0 = \lim_{x_{n+1} \searrow 0} u \quad \text{and} \quad U_1 = \lim_{x_{n+1} \searrow 0} \partial_{n+1} u$$

in terms of the action of the components of the associated Calderón projector. This part of the analysis results in the set of equations (3.3), and it is an adaptation of the development in Hörmander [6, pp. 234–236]. Theorem 2.9 is then used to estimate the domain of analytic continuation of some of the terms in (3.3). We use the notation of Section 1. Also, δ is the Dirac measure and δ' is its derivative. Finally, $\gamma_0^+ v$ is the trace $\lim_{x_{n+1} \searrow 0} v$.

Let u° be the extension by zero of u from $\tau(\overline{a}) \times [0, \lambda]$ to \mathbb{R}^{n+1} , and write $U_0 = u^\circ(\cdot, 0)$ and $U_1 = (\partial_{n+1} u^\circ)(\cdot, 0)$. Fix $R > 2k/(1 - \varepsilon^2)$, and let χ be an element of $C^\infty(\mathbb{R}^{n+1})$ satisfying

$$\chi(\xi) = \begin{cases} 0, & |\xi| \leq \sqrt{1 - \varepsilon^2} R/2, \\ 1, & |\xi| \geq \sqrt{1 - \varepsilon^2} R, \end{cases} \quad \xi \in \mathbb{R}^{n+1}.$$

The function $\chi - 1$ is an element of the space $S^{-\infty}(\mathbb{R}^{n+1})$ of rapidly decreasing symbols. We readily find by iteration that, for each $\alpha \in \mathbb{N}_0^{n+1}$, $\partial^\alpha (k^2 - |\xi|^2)^{-1} = O(|\xi|^{-2-|\alpha|})$ as $|\xi| \rightarrow \infty$, so the function q , defined by

$$q(\xi) = \frac{\chi(\xi)}{k^2 - |\xi|^2}, \quad \xi \in \mathbb{R}^{n+1},$$

is an element of the space $S^{-2}(\mathbb{R}^{n+1})$. The corresponding pseudodifferential operator $Q = \text{OPS}(q)$ is a left parametrix of the Helmholtz operator on \mathbb{R}^{n+1} , in that the symbol of the compound $Q(\Delta + k^2)$ is the function χ (the symbol of the compound actually *is* χ and is not merely asymptotically equivalent to χ , since the symbol $k^2 - |\xi|^2$ of the Helmholtz operator depends only on the cotangent variables ξ_1, \dots, ξ_{n+1} .) In the sense of distributions in $\mathcal{D}'(\mathbb{R}^{n+1})$, we have

$$\begin{aligned} (\Delta + k^2)u^\circ &= \sum_{j=1}^n (u^\circ|_{x_j=-a_j} \otimes \delta'(x_j + a_j) + u^\circ|_{x_j=a_j} \otimes \delta'(x_j - a_j)) \\ (3.1) \quad &+ \sum_{j=1}^n ((\partial_j u^\circ)|_{x_j=-a_j} \otimes \delta(x_j + a_j) + (\partial_j u^\circ)|_{x_j=a_j} \otimes \delta(x_j - a_j)) \\ &+ U_0 \otimes \delta'(x_{n+1}) + u^\circ|_{x_{n+1}=\lambda} \otimes \delta'(x_{n+1} - \lambda) \\ &+ U_1 \otimes \delta(x_{n+1}) + (\partial_{n+1} u^\circ)|_{x_{n+1}=\lambda} \otimes \delta(x_{n+1} - \lambda). \end{aligned}$$

Applying the compound $\gamma_0^+ Q$ to both sides of (3.1) gives

$$(3.2) \quad (I - \Pi_{00}^+) U_0 + \gamma_0^+ \text{OPS}(\chi - 1) u^\circ = \Pi_{01}^+ U_1 + S_1 + S_2,$$

where the operators

$$\Pi_{00}^+ = \gamma_0^+ Q((\cdot) \otimes \delta'(x_{n+1})) \quad \text{and} \quad \Pi_{01}^+ = \gamma_0^+ Q((\cdot) \otimes \delta(x_{n+1}))$$

are components of the Calderón projector associated with the setup of Theorem 1.1, and where

$$S_1 = \sum_{j=1}^n [\gamma_0^+ Q(u^\circ|_{x_j=-a_j} \otimes \delta'(x_j + a_j)) + \gamma_0^+ Q((\partial_j u^\circ)|_{x_j=-a_j} \otimes \delta(x_j + a_j))]$$

as well as

$$S_2 = \sum_{j=1}^n [\gamma_0^+ Q(u^\circ|_{x_j=a_j} \otimes \delta'(x_j - a_j)) + \gamma_0^+ Q((\partial_j u^\circ)|_{x_j=a_j} \otimes \delta(x_j - a_j))] \\ + \gamma_0^+ Q(u^\circ|_{x_{n+1}=\lambda} \otimes \delta'(x_{n+1} - \lambda)) + \gamma_0^+ Q((\partial_j u^\circ)|_{x_{n+1}=\lambda} \otimes \delta(x_{n+1} - \lambda)).$$

If $I - \Pi_{00}^+$ and Π_{01}^+ have inverses, then (3.2) implies

$$(3.3) \quad \begin{cases} U_0 = (I - \Pi_{00}^+)^{-1} \Pi_{01}^+ U_1 \\ \quad + (I - \Pi_{00}^+)^{-1} (S_1 + S_2) + (I - \Pi_{00}^+)^{-1} \gamma_0^+ \text{OPS}(1 - \chi) u^\circ, \\ U_1 = (\Pi_{01}^+)^{-1} (I - \Pi_{00}^+) U_0 \\ \quad - (\Pi_{01}^+)^{-1} (S_1 + S_2) + (\Pi_{01}^+)^{-1} \gamma_0^+ \text{OPS}(\chi - 1) u^\circ. \end{cases}$$

We shall use (3.3) to relate the domains of analytic continuability of the Dirichlet datum U_0 and the Neumann datum U_1 . To this end, we first show that the inverses $(I - \Pi_{00}^+)^{-1}$ and $(\Pi_{01}^+)^{-1}$ exist and have suitable real-analytic symbols. Fix $j \in \{1, \dots, n+1\}$, $l \in \{0, 1\}$ and $C \in \mathbb{R}$.

Lemma 3.1. *For each fixed $x_j \in \mathbb{R}$, the operator $Q((\cdot) \otimes \delta^{(l)}(x_j - C))$ has a symbol $\pi_{l,j}(\cdot, x_j - C) \in S_{r-a}^{l-1, \varepsilon}$. The function $\pi_{l,j}(\xi_{(j)}, s)$ is analytically continuable in s to the domain $\{s \in \mathbb{C}, \text{Re } s \neq 0\}$.*

Proof. Following Hörmander [6, p. 109], we will consider a function $\phi \in C_0^\infty(]-1, 1[)$ satisfying $\int_{t \in \mathbb{R}} \phi(t) dt = 1$. If $V \in C_0^\infty(\mathbb{R}^n)$, then

$$Q(V \otimes \delta^{(l)}(x_j - C))(x) = \lim_{\kappa \rightarrow 0} Q(V \kappa^{-1} \phi^{(l)}((x_j - C)/\kappa))(x) \\ = (2\pi)^{-n} \int_{\xi_{(j)} \in \mathbb{R}^n} e^{i\xi_{(j)} x_{(j)}} \widehat{V}(\xi_{(j)}) \pi_{l,j}(\xi_{(j)}, x_j - C),$$

with the one-parameter family of symbols $\pi_{l,j}(\cdot, s)$ given by

$$\pi_{l,j}(\xi_{(j)}, s) = \frac{1}{2\pi} \lim_{\kappa \rightarrow 0} \int_{t \in \mathbb{R}} (it)^l q(\xi_{(j)}, t) e^{its} \widehat{\phi}(\kappa t), \quad \xi_{(j)} \in \mathbb{R}^n, s \in \mathbb{R}.$$

Writing $\rho = \sqrt{1 - \varepsilon^2} R$ and $\tilde{q}(\xi_{(j)}, t) = (k^2 - |\xi_{(j)}|^2 - t^2)^{-1}$, we get, for $s \neq 0$,

$$\int_{t \in \mathbb{R}} t^l q(\xi_{(j)}, t) e^{its} \widehat{\phi}(\kappa t) dt = \int_{|t| > \rho} \tilde{q}(\xi_{(j)}, t) t^l e^{its} \widehat{\phi}(\kappa t) dt + \int_{|t| \leq \rho} t^l q(\xi_{(j)}, t) e^{its} \widehat{\phi}(\kappa t) dt.$$

(Recall that $\chi(\xi_{(j)}, t) = 1$ and thus $q(\xi_{(j)}, t) = \tilde{q}(\xi_{(j)}, t)$ when $|\xi_{(j)}|^2 + t^2 > \rho^2$.) The function $\mathbb{C} \ni t \mapsto t^l \tilde{q}(\xi_{(j)}, t) e^{it(x_j - d)} \widehat{\phi}(\kappa t)$ is analytic in $\{t \in \mathbb{C}, |t| > \rho\}$ and exponentially decreasing w.r.t. $|t|$ when $\text{sgn } \text{Im } t = \text{sgn } s$, so

$$\int_{|t| > \rho} \tilde{q}(\xi_{(j)}, t) t^l e^{its} \widehat{\phi}(\kappa t) dt = - \int_{\phi = \pi \text{sgn } s}^0 \tilde{q}(\xi_{(j)}, \rho e^{i\phi}) i(\rho e^{i\phi})^{l+1} e^{i\rho \exp(i\phi)s} \widehat{\phi}(\kappa \rho e^{i\phi}),$$

and consequently

$$2\pi i^{-l} \pi_{l,j}(\xi_{(j)}, s) = \int_{|t| \leq \rho} t^l q(\xi_{(j)}, t) e^{its} dt + \int_{\phi=0}^{\pi \text{sgn } s} \tilde{q}(\xi_{(j)}, \rho e^{i\phi}) i(\rho e^{i\phi})^{l+1} e^{i\rho \exp(i\phi)s} d\phi.$$

In particular, the mapping $\mathbb{R} \setminus \{0\} \ni s \mapsto \pi_{l,j}(\xi_{(j)}, s)$ is analytically continuable to the complex plane without the imaginary axis. Also, for $|\xi_{(j)}| > \rho$ and $s \in \mathbb{R} \setminus \{0\}$ we have

$$(3.4) \quad \begin{aligned} 2\pi i^{-l} \pi_{l,j}(\xi_{(j)}, s) &= 2\pi i (\operatorname{sgn} s) \sum_t^{\operatorname{sgn} s} \operatorname{Res} \tilde{q}(\xi_{(j)}, t) t^l e^{its} \\ &= -\frac{\pi((\operatorname{sgn} s) i \sqrt{|\xi_{(j)}|^2 - k^2})^l e^{-|s| \sqrt{|\xi_{(j)}|^2 - k^2}}}{\sqrt{|\xi_{(j)}|^2 - k^2}}, \end{aligned}$$

where \sum_t^+ Res and \sum_t^- Res signify the sum of residues w.r.t. t in the upper and in the lower complex half-plane, respectively. Thus, for nonzero real s , the symbol $\pi_{l,j}(\cdot, s)$ is in $S^{-\infty}$. There is a jump discontinuity in (3.4) across $s = 0$, but we readily find that both $\lim_{s \nearrow 0} \pi_{l,j}(\cdot, s)$ and $\lim_{s \searrow 0} \pi_{l,j}(\cdot, s)$ are in S^{l-1} . Also, if the real part of a component of $\zeta_{(j)} \in K_\varepsilon^n$ is greater than R , then $|\operatorname{Re} \zeta_{(j)}| > R$ and

$$\operatorname{Re}[(\zeta_{(j)})^2] - k^2 > (1 - \varepsilon^2)|\operatorname{Re} \zeta_{(j)}|^2 - k^2 > (1 - \varepsilon^2)R^2 - k^2 > 0,$$

so the right-hand side in (3.4) is analytic on K_ε^n . Finally, for such $\zeta_{(j)}$, we have

$$\begin{aligned} \left| [(\zeta_{(j)})^2 - k^2]^{1/2} \right| &= \sqrt{|\operatorname{Re} \zeta_{(j)}|^2 - |\operatorname{Im} \zeta_{(j)}|^2 - k^2 + 2i \operatorname{Re} \zeta_{(j)} \cdot \operatorname{Im} \zeta_{(j)}} \\ &\geq \sqrt{|\operatorname{Re} \zeta_{(j)}|^2 (1 - \varepsilon^2) - k^2} \geq C(1 + |\operatorname{Re} \zeta_{(j)}|) \end{aligned}$$

when

$$C \leq \frac{k\sqrt{3(1 - \varepsilon^2)}}{2k + \sqrt{1 - \varepsilon^2}},$$

so $\pi_{l,j}(\cdot, s) \in S_{r-a}^{l-1, \varepsilon}$ for every real s . \square

Corollary 3.2. *The symbol of the operator $\gamma_0^+ Q((\cdot) \otimes \delta^{(l)}(x_{n+1} - C))$ is the function $\lim_{x_{n+1} \searrow 0} \pi_{1,n+1}(\cdot, x_{n+1} - C) \in S_{r-a}^{l-1, \varepsilon}$. In particular, $I - \Pi_{00}^+$ and Π_{01}^+ have symbols*

$$1 - \lim_{x_{n+1} \searrow 0} \pi_{1,n+1}(\cdot, x_{n+1}) \in S_{r-a}^{0, \varepsilon} \quad \text{and} \quad \lim_{x_{n+1} \searrow 0} \pi_{0,n+1}(\cdot, x_{n+1}) \in S_{r-a}^{-1, \varepsilon},$$

respectively.

Lemma 3.3. *$I - \Pi_{00}^+$ and Π_{01}^+ have inverses with symbols in $S_{r-a}^{0, \varepsilon}$ and $S_{r-a}^{1, \varepsilon}$, respectively.*

Proof. For $\xi' \in \mathbb{R}^n$ with $|\xi'| > \sqrt{1 - \varepsilon^2}R$, we have

$$1 - \lim_{x_{n+1} \searrow 0} \pi_{1,n+1}(\xi', x_{n+1}) = \frac{1}{2} \quad \text{and} \quad \lim_{x_{n+1} \searrow 0} \pi_{0,n+1}(\xi', x_{n+1}) = -\frac{1}{2\sqrt{|\xi'|^2 - k^2}}.$$

Thus, the symbols $\mathbb{R}^n \ni \xi' \mapsto 2\chi(\xi', 0)$ and $\mathbb{R}^n \ni \xi' \mapsto -2\sqrt{|\xi'|^2 - k^2}\chi(\xi', 0)$ define inverses $(I - \Pi_{00}^+)^{-1}$ and $(\Pi_{01}^+)^{-1}$, respectively. Trivially, the symbol of $(I - \Pi_{00}^+)^{-1}$ is in $S_{r-a}^{0, \varepsilon}$. Also, if the real part of a component of $\zeta' \in K_\varepsilon^n$ is greater than R , then

$$\begin{aligned} \left| ((\zeta')^2 - k^2)^{1/2} \right| &= \sqrt{|\operatorname{Re} \zeta'|^2 - |\operatorname{Im} \zeta'|^2 - k^2 + 2i \operatorname{Re} \zeta' \cdot \operatorname{Im} \zeta'} \\ &\leq \sqrt{|\operatorname{Re} \zeta'|^2 + 2i \operatorname{Re} \zeta' \cdot \operatorname{Im} \zeta'} \leq \sqrt{1 + 2i\varepsilon} |\operatorname{Re} \zeta| < 5^{1/4}(1 + |\operatorname{Re} \zeta|), \end{aligned}$$

and in conclusion the symbol of $(\Pi_{01}^+)^{-1}$ is in $S_{r-a}^{1, \varepsilon}$. \square

We next estimate the analytic continuability of terms of the forms

$$(3.5) \quad (I - \Pi_{00}^+)^{-1} \gamma_0^+ Q(u^\circ|_{x_j=C} \otimes \delta^{(l)}(x_j - C)), \quad (\Pi_{01}^+)^{-1} \gamma_0^+ Q(u^\circ|_{x_j=C} \otimes \delta^{(l)}(x_j - C))$$

occurring in (3.3). First assume $j = n + 1$. Remark 2.10, Corollary 3.2 and Lemma 3.3 imply that the compounds $(I - \Pi_{00}^+)^{-1} \gamma_0^+ Q((\cdot) \otimes \delta^{(l)}(x_j - C))$ and $(\Pi_{01}^+)^{-1} \gamma_0^+ Q((\cdot) \otimes \delta^{(l)}(x_j - C))$ have symbols in $S_{r-a}^{l-1, \varepsilon}$ and $S_{r-a}^{l, \varepsilon}$, respectively. Thus, by Theorem 2.9, the functions in (3.5) are analytically continuable to each complex polyrectangle $T(\tilde{a}, \tilde{b}^\pm)$ with $\tilde{a}_j \in]0, a_j[$ and $\tilde{b}_j^\pm \in]0, \varepsilon \min\{a_j - \tilde{a}_j, b_j^\pm\}[$, $j = 1, \dots, n$.

Now assume $j \neq n + 1$ and let d be the symbol of $(I - \Pi_{00}^+)^{-1}$ or of $(\Pi_{01}^+)^{-1}$. Abbreviate $\xi' = \xi_{(n+1)}$, $\xi'' = \xi_{(j)}$, $\xi''' = \xi_{(j, n+1)}$, and similarly for other vectors. For $x' = x_{(n+1)} \in \mathbb{R}^n$ we have

$$\begin{aligned} & (2\pi)^{2n} \text{OPS}(d) \gamma_0^+ Q(u^\circ|_{x_j=C} \otimes \delta^{(l)}(x_j - C))(x') \\ &= \iiint_{w', \eta', y'', \xi'' \in \mathbb{R}^n} e^{i\eta'(x' - w') + i\xi''' w''' - i\xi'' y''} d(\eta') \pi_{l,j}(\xi'', x_j - C) u^\circ|_{y_j=C}(y'') \\ &= 2\pi \iiint_{\eta', y'', \xi'' \in \mathbb{R}^n, w_j \in \mathbb{R}} e^{i\eta_j(x_j - w_j) + i\eta''' x''' - i\xi'' y''} d(\eta') \pi_{l,j}(\xi'', x_j - C) u^\circ|_{y_j=C}(y'') \delta(\xi''' - \eta''') \\ &= (2\pi)^2 \iiint_{y'', \xi'' \in \mathbb{R}^n, \eta_j \in \mathbb{R}} e^{i\eta_j x_j + i\xi''' x''' - i\xi'' y''} d(\xi''', \eta_j) \pi_{l,j}(\xi'', x_j - C) u^\circ|_{y_j=C}(y'') \delta(\eta_j) \\ &= (2\pi)^{n+2} \left[\gamma_0^+ \text{OPS}(d(\xi''', 0) \pi_{l,j}(\xi'', x_j - C)) u^\circ|_{x_j=C} \right](x'). \end{aligned}$$

By Remark 2.10, Lemma 3.1 and Lemma 3.3, for each fixed x_j the symbol $\mathbb{R}^n \ni \xi'' \mapsto d(\xi''', 0) \pi_{l,j}(\xi'', x_j - C)$ is in $S_{r-a}^{l, \varepsilon}$, and it is analytically continuable to $\{x_j \in \mathbb{C}, \text{Re } x_j \neq C\}$. Finally, using Theorem 2.9, we obtain at least the same estimates on the analytic continuability of the terms (3.5) as in the case $j = n + 1$.

It remains to estimate the analytic continuability of the terms

$$(3.6) \quad (I - \Pi_{00}^+)^{-1} \gamma_0^+ \text{OPS}(1 - \chi) u^\circ \quad \text{and} \quad (\Pi_{01}^+)^{-1} \gamma_0^+ \text{OPS}(\chi - 1) u^\circ$$

occurring in (3.3). Again write d for the symbol of $(I - \Pi_{00}^+)^{-1}$ or of $(\Pi_{01}^+)^{-1}$. We readily find that $\text{OPS}(d) \gamma_0^+ \text{OPS}(\chi - 1) u^\circ = \gamma_0^+ \text{OPS}(d(\xi')(\chi(\xi) - 1)) u^\circ$. The symbol $d(\xi')(\chi(\xi) - 1)$ is in $C_0^\infty(\mathbb{R}^{n+1})$, and the function u° is bounded and compactly supported in \mathbb{R}^{n+1} . The Fourier transform \widehat{u}° of u° is in $C^\infty(\mathbb{R}^{n+1})$, so the function $d \cdot (\chi - 1) \cdot \widehat{u}^\circ$ is smooth and compactly supported. The Fourier-Laplace transform v of $d \cdot (\chi - 1) \cdot \widehat{u}^\circ$ is then an entire function on \mathbb{C}^{n+1} . But for every x in \mathbb{R}^{n+1} ,

$$(2\pi)^{n+1} \text{OPS}(d(\xi')(\chi(\xi) - 1)) u^\circ(x) = \int_{\mathbb{R}^{n+1}} e^{ix\xi} d(\xi')(\chi(\xi) - 1) \widehat{u}^\circ(\xi) d\xi = v(-x),$$

so the function $\text{OPS}(d(\xi')(\chi(\xi) - 1)) u^\circ$ is analytically continuable to the whole complex space \mathbb{C}^{n+1} . In particular, its restriction $\gamma_0^+ \text{OPS}(d(\xi')(\chi(\xi) - 1)) u^\circ$ is analytically continuable to the 'polystrip' $T(a', \pm\infty)$.

Theorem 1.1 now follows from (3.3) and the fact that the constant ε can be chosen arbitrarily close to 1 in the analysis in this section.

4. OUTLOOK

We expect that the method of proof of Theorem 1.1 presented here can be extended to handle general linear, second-order, analytic, elliptic differential operators in two or more independent variables, as well as general analytic pieces of the boundary. This requires an analysis similar to that of Section 2, but done on a subset of standard pseudodifferential operators with symbols that are real-analytic and dependent on the full variable set (x, ξ) .

REFERENCES

1. L. Boutet de Monvel, *Opérateurs pseudo-différentiels analytiques et opérateurs d'ordre infini*, Ann. I. Fourier **22** (1972), 229–268.
2. A. Doicu, Yu. Eremin, and T. Wriedt, *Acoustic & Electromagnetic Scattering Analysis Using Discrete Sources*, Academic Press, 2000.
3. Graeme Fairweather and Andreas Karageorghis, *The method of fundamental solutions for elliptic boundary value problems*, Advances in Computational Mathematics **9** (1998), 69–95.
4. Gerald B. Folland, *Real Analysis – Modern Techniques and Their Applications*, John Wiley & Sons, 1984.
5. Lars Hörmander, *The Analysis of Linear Partial Differential Operators I*, Grundlehren der mathematischen Wissenschaften, vol. 256, Springer Verlag, 1983.
6. ———, *The Analysis of Linear Partial Differential Operators III*, Grundlehren der mathematischen Wissenschaften, vol. 274, Springer Verlag, 1985.
7. R. Kangro and U. Kangro, *Extendability of Solutions of Helmholtz's Equation across a hyperplane in Three Dimensions*, Proc. of the International Conf. on Computational and Mathematics Methods in Science and Engineering, CMMSE 2006 (Madrid), 21–25 September 2006, pp. 433–437.
8. M. Karamehmedović, P.-E. Hansen, K. Dirscherl, E. Karamehmedović, and T. Wriedt, *Profile estimation for Pt submicron wire on rough Si substrate from experimental data*, Opt. Express **20** (2012), 21678–21686.
9. M. Karamehmedović, P.-E. Hansen, and T. Wriedt, *An efficient scattering model for PEC and penetrable nanowires on a dielectric substrate*, J. Eur. Opt. Soc. Rapid **6** (2011), 11021.
10. ———, *A fast inversion method for highly conductive submicron wires on a substrate*, J. Eur. Opt. Soc. Rapid **6** (2011), 11039.
11. M. Karamehmedović, M.-P. Sørensen, P.-E. Hansen, and A. V. Lavrinenko, *Application of the method of auxiliary sources to a defect-detection inverse problem of optical diffraction microscopy*, J. Eur. Opt. Soc. Rapid **5** (2010), 10021.
12. G. Köthe, *Topological Vector Spaces II*, Grundlehren der mathematischen Wissenschaften, vol. 237, Springer, 1979.
13. J. M. Lee, *Introduction to Smooth Manifolds*, Springer, 2003.
14. R. F. Millar, *The Location of Singularities of Two-Dimensional Harmonic Functions. I: Theory*, SIAM J. Math. Anal. **1** (1970), 333–344.
15. ———, *The Location of Singularities of Two-Dimensional Harmonic Functions. II: Applications*, SIAM J. Math. Anal. **1** (1970), 345–353.
16. ———, *Singularities of Solutions to Linear, Second Order Analytic Elliptic Equations in Two Independent Variables I. The Completely Regular Boundary*, Appl. Anal. **1** (1971), 101–121.
17. ———, *Singularities of two-dimensional exterior solutions of the Helmholtz equation*, Proc. Camb. Phil. Soc. **69** (1971), 175–188.
18. ———, *Singularities of Solutions to Linear, Second Order, Analytic Elliptic Equations in Two Independent Variables. II The Piecewise Regular Boundary*, Appl. Anal. **1** (1972), 301–320.
19. ———, *Singularities of solutions to exterior analytic boundary value problems for the Helmholtz equation in three independent variables. I: The plane boundary*, SIAM J. Math. Anal. **7** (1976), no. 1, 131–156.
20. ———, *Singularities of solutions to exterior analytic boundary value problems for the Helmholtz equation in three independent variables. II: The axisymmetric boundary*, SIAM J. Math. Anal. **10** (1979), no. 4, 682–694.
21. ———, *The Analytic Continuation of Solutions to Elliptic Boundary Value Problems in Two Independent Variables*, J. Math. Anal. Appl. **76** (1980), 498–515.
22. ———, *Singularities and the Rayleigh Hypothesis for Solutions to the Helmholtz Equation*, IMA J. Appl. Math. **37** (1986), 155–171.
23. U. Kangro R. Kangro and R. Nicolaides, *Extendability of solutions of Helmholtz's equation to the interior of a two-dimensional scatterer*, Q. Appl. Math. **LVIII** (2000), no. 3, 591–600.
24. B. Sternin and V. Shatalov, *Differential Equations on Complex Manifolds*, Mathematics and Its Applications, vol. 276, Kluwer Academic Publishers, 1994.
25. B. Yu. Sternin and V. E. Shatalov, *Analytic Continuation of Solutions of Integral Equations and Localization of Singularities*, Diff. Equat.+ **32** (1996), no. 11, 1541–1549.

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